Directed equality with dinaturality

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Motivation: Directed type theory

Type theories with refl$/J$ *are intrinsically about symmetric equality.*
Directed type theory *is the generalization to "directed equality".*

The interpretation of directed type theory with *(1-)categories*:

$$
\begin{aligned}
\text{Types} &\rightsquigarrow \text{Categories} \\
\text{Terms} &\rightsquigarrow \text{Functors} \\
\text{Points of a type} &\rightsquigarrow \text{Objects of a category} \\
\text{Equalities } e : a = b &\rightsquigarrow \text{Morphisms } e : \hom(a, b) \\
=_A : A \times A \to \text{Type} &\rightsquigarrow \hom_{\mathbb{C}} : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbf{Set}
\end{aligned}
$$

$\rightarrow$ Now types have a *polarity*, $\mathbb{C}$ and $\mathbb{C}^{\mathsf{op}}$, i.e., the opposite category.
$\rightarrow$ Now equalities $e : \hom(a, b)$ have *directionality.*

Current approaches to directed type theory

- Semantically, refl should be $\mathrm{id}_c \in \hom_{\mathbb{C}}(c, c)$ for $c : \mathbb{C}$.
- Transitivity of directed equality $\rightsquigarrow$ composition of morphisms in $\mathbb{C}$.

$$\dfrac{\dfrac{}{[z : \mathbb{C}^{\mathsf{op}}, c : \mathbb{C}] \quad \hom(z, c) \vdash \hom(z, c)}\,(\mathsf{id})}{[a : \mathbb{C}^{\mathsf{op}}, b : \mathbb{C}, c : \mathbb{C}] \ \hom(a, b),\ \hom(\bar{b}, c) \vdash \hom(a, c)}\,(\mathsf{J})$$

- However, directed type theory is not so straightforward:

$$\dfrac{a : \mathbb{C}}{\mathsf{refl}_a \ldots ? : \hom_{\mathbb{C}}(a, a)} \quad \rightsquigarrow \quad \dfrac{a : \mathbb{C}^{\mathsf{core}}}{\mathsf{refl}_a : \hom(\mathsf{i}^{\mathsf{op}}(a), \mathsf{i}(a))} \quad \text{[North 2018]}$$

- *Problem:* rule is not functorial w.r.t. variance of $\hom_{\mathbb{C}} : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbf{Set}$, since $a : \mathbb{C}$ appears both contravariantly and covariantly.
- A possible approach to DTT in **Cat**: use groupoids!
  $\rightarrow$ Use the maximal subgroupoid $\mathbb{C}^{\mathsf{core}}$ to collapse the two variances.
- Then a J-like rule is validated, but *again using groupoidal structure*.

Dinatural directed first-order type theory

We show a **first-order** non-dependent directed type theory, with semantics:

Syntax	$\rightsquigarrow$	Semantics
Types	$\rightsquigarrow$	Categories
Contexts	$\rightsquigarrow$	Product of categories
Terms	$\rightsquigarrow$	Functors $F : \mathbb{C} \to \mathbb{D}$
Predicates	$\rightsquigarrow$	Dipresheaves, i.e., functors $P : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbf{Set}$
	$\rightsquigarrow$	e.g., $\hom$-functors $\mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbf{Set}$
Entailments	$\rightsquigarrow$	Dinatural transformations (not required to compose)
Quantifiers	$\rightsquigarrow$	Ends $\displaystyle\int_{x:\mathbb{C}} P(\overline{x}, x)$, coends $\displaystyle\int^{x:\mathbb{C}} P(\overline{x}, x)$.

- Dinaturality solves the variance issue without groupoids, and tells what syntactic restriction to put on J to avoid symmetry.
- We give "logical rules" to (co)ends as the *directed quantifiers* of DTT: $\rightsquigarrow$ rules of DTT give *simple proofs* in category theory, with $\hom$ as $=$.
- We do first-order because (co)end calculus is typically first-order.

Syntax – judgements for types

- Judgement $\boxed{C\ \mathsf{type}}$ for types:

$$\frac{C\ \mathsf{type}}{C^{\mathsf{op}}\ \mathsf{type}} \qquad \frac{C\ \mathsf{type} \qquad D\ \mathsf{type}}{C \times D\ \mathsf{type}} \qquad \frac{C\ \mathsf{type} \qquad D\ \mathsf{type}}{[C, D]\ \mathsf{type}} \qquad \frac{}{\top\ \mathsf{type}}$$

- **Semantics:** C type is interpreted by a category $[\![C]\!]$.
- Definitional equality on types $\boxed{C = C'\ \mathsf{type}}$ is such that

$$\begin{aligned} (C^{\mathsf{op}})^{\mathsf{op}} &= C \\ (C \times D)^{\mathsf{op}} &= C^{\mathsf{op}} \times D^{\mathsf{op}} \\ ([C, D])^{\mathsf{op}} &= [C^{\mathsf{op}}, D^{\mathsf{op}}] \\ (\top)^{\mathsf{op}} &= \top \end{aligned}$$

- A judgement $\boxed{\Gamma\ \mathsf{ctx}}$ for contexts, i.e., lists of types, with also Γ^{op} ctx.
- **Semantics:** contexts are interpreted as the product of categories.

$$[\![\Gamma := [C_1, \dots C_n]]\!] := [\![C_1]\!] \times \cdots \times [\![C_n]\!]$$

Directed type theory: judgements for terms

- A judgement $\boxed{\Gamma \vdash t : C}$ for simply-typed terms.
- **Semantics:** terms are interpreted as functors $[\![t]\!] : [\![\Gamma]\!] \to [\![C]\!]$.

$$\frac{\Gamma \ni x : C}{\Gamma \vdash x : C} \qquad \frac{}{\Gamma \vdash ! : \top} \qquad \frac{\Gamma \vdash s : C \qquad \Gamma \vdash t : D}{\Gamma \vdash \langle s, t \rangle : C \times D}$$

$$\frac{\Gamma \vdash p : C \times D}{\Gamma \vdash \pi_1(p) : C} \qquad \frac{\Gamma \vdash p : C \times D}{\Gamma \vdash \pi_2(p) : D} \qquad \dots$$

$$\frac{\Gamma \vdash t : C}{\Gamma^{\mathsf{op}} \vdash t^{\mathsf{op}} : C^{\mathsf{op}}}$$

- Definitional equality on terms $\boxed{\Gamma \vdash t = t' : C}$ is such that $(t^{\mathsf{op}})^{\mathsf{op}} = t$.

Syntax – predicates

- A judgement $\boxed{[\Gamma]\ P\ \mathsf{prop}}$ for predicates.
- **Semantics:** dipresheaves, i.e., functors $[\![P]\!] : [\![\Gamma]\!]^{\mathsf{op}} \times [\![\Gamma]\!] \to \mathbf{Set}$.
- Formation rules:

$$\frac{[\Gamma]\ P\ \mathsf{prop} \qquad [\Gamma]\ Q\ \mathsf{prop}}{[\Gamma]\ P \times Q\ \mathsf{prop}} \quad \frac{[\Gamma]\ P\ \mathsf{prop} \qquad [\Gamma]\ Q\ \mathsf{prop}}{[\Gamma]\ P \Rightarrow Q\ \mathsf{prop}} \quad \frac{}{[\Gamma]\ \top\ \mathsf{prop}}$$

$$\frac{[\Gamma, x : C]\ P(x)\ \mathsf{prop}}{[\Gamma]\ \int^{x:C} P(x)\ \mathsf{prop}} \quad \frac{[\Gamma, x : C]\ P(x)\ \mathsf{prop}}{[\Gamma]\ \int_{x:C} P(x)\ \mathsf{prop}}$$

- **Semantics:** $\times$ is the pointwise product of dipresheaves in **Set**, $\Rightarrow$ is the pointwise $\hom$ in **Set**, (co)ends are always taken in **Set**.

Syntax – predicates (contd.)

- Directed equality predicates:

$$\frac{\Gamma^{\mathsf{op}}, \Gamma \vdash s : C^{\mathsf{op}} \qquad \Gamma^{\mathsf{op}}, \Gamma \vdash t : C}{[\Gamma] \;\; \hom_C(s,t) \ \mathsf{prop}}$$

- **Key idea:** I can use variables from Γ or from Γ^{op} in the terms s, t.
- We indicate with $\overline{x} : C^{\mathsf{op}}$ when variables are taken from Γ^{op}.
- This is what allows us to *write* these entailments:

$$\begin{array}{rrl} [x : C] & \Phi \vdash \mathsf{refl} & : \hom(\overline{x}, x) \\ [a : C^{\mathsf{op}}, b : C, c : C] \; \hom(a,b), & \hom(\overline{b}, c), \Phi \vdash \mathsf{trans} & : \hom(a,c) \\ [a : C^{\mathsf{op}}, b : C] & \hom(a,b), \Phi \vdash \mathsf{sym} & : \hom(\overline{b}, \overline{a}) \end{array}$$

- **Polarity of a position**: *positive* when taken from Γ, *negative* when Γ^{op}.
- **Variance of a variable**:
  natural when always taken from Γ,
  dinatural (i.e., mixed-variance) when sometimes from Γ, sometimes Γ^{op}.

Syntax – entailments

- A judgement $\boxed{[\Gamma]\ \Phi \vdash \alpha : P}$ for entailments (Φ is a list of predicates).

$$[x : C, y : D, \Gamma]\ \Phi(\overline{x}, x, \overline{y}, y, \dots) \vdash \alpha : P(\overline{x}, x, \overline{y}, y, \dots)$$

- **Semantics:** interpreted as dinatural transformations $[\![\alpha]\!] : [\![\Phi]\!] \xrightarrow{\bullet\bullet} [\![P]\!]$:

$$\forall x \in [\![\Gamma]\!], \alpha_x : [\![\Phi]\!](x, x) \longrightarrow [\![P]\!](x, x)$$

- Dinaturals do not always compose; they do with *natural* transformations.

$$\frac{P \longrightarrow Q \xrightarrow{\bullet\bullet} R \longrightarrow T}{P \xrightarrow{\bullet\bullet} T}$$

- We capture left/right cut rules with naturals, e.g.: nat on the right:

$$\frac{\begin{array}{r} P, Q \text{ do not depend on } \Gamma \\ [z : C, \Gamma]\ \Phi(\overline{z}, z) \vdash \gamma \quad\ : P(\overline{z}, z) \\ [a : C^{\mathsf{op}}, b : C, \Gamma]\ k : P(a, b), \Phi(\overline{a}, \overline{b}) \vdash \alpha[k] : Q(a, b) \end{array}}{[z : C, \Gamma]\ \Phi(\overline{z}, z) \vdash \alpha[\gamma] : Q(\overline{z}, z)}\ \text{(cut-nat)}$$

Takeaway: whenever we need dinats to compose, they do because of this.

Syntax – rules for $\hom$

- Directed equality introduction:

$$\frac{}{[x : C, \Gamma]\ \Phi \vdash \mathsf{refl}_x : \hom_C(\overline{x}, x)}\ (\mathsf{refl})$$

- **Semantics:** refl is validated precisely by identity morphisms in $[\![C]\!]$.
- Directed equality elimination:

$$\frac{[z : C, \Gamma] \qquad\qquad \Phi(z, \overline{z}) \vdash h : P(\overline{z}, z)}{[a : C^{\mathsf{op}}, b : C, \Gamma]\ e : \hom_C(a, b), \Phi(\overline{a}, \overline{b}) \vdash J(h) : P(a, b)}\ (J)$$

If I have a directed equality $e : \hom_C(a, b)$ in context,

- I can contract it only if a, b appear *only positively* in the conclusion P,
- and a, b appear *only negatively* in the context Φ.

- Then, it is enough to prove that P holds “on the diagonal” $z : C$.

- **Semantics:** functoriality of $[\![\Phi]\!]$ and $[\![P]\!]$.

Directed type theory with dinaturality – examples

Example (Transitivity of directed equality)

Composition is natural in $a : C^{\mathsf{op}}, c : C$ and dinatural in $b : C$:

$$\dfrac{\dfrac{}{[z : C, c : C] \quad g : \hom(\overline{z}, c) \vdash g : \hom(\overline{z}, c)}\,(\mathsf{var})}{[a : C^{\mathsf{op}}, b : C, c : C] \; f : \hom(a, b),\; g : \hom(\overline{b}, c) \vdash J(g) : \hom(a, c)}\,(J)$$

We contract $f : \hom(a, b)$. Rule (J) can be applied: a, b appear only negatively in ctx (a does not) and positively in conclusion ($\overline{b}$ does not).

Directed type theory with dinaturality – examples

Example (Congruence)

Functoriality of terms P is natural in $a : C^{\mathsf{op}}, b : C$ for terms $C \vdash F : D$:

$$\dfrac{\dfrac{\dfrac{}{[z : D]\ \cdot \vdash \mathsf{refl}_x : \hom_D(\overline{x}, x)}\ (\mathsf{refl})}{[z : C]\ \cdot \vdash F^*(\mathsf{refl}_x) : \hom_D(F(\overline{z}), F(z))}\ (\mathsf{idx})}{[a : C^{\mathsf{op}}, b : C]\ e : \hom_C(a, b) \vdash J(F^*(\mathsf{refl}_x)) : \hom_D(F(a), F(b))}\ (J)$$

Example (Transport)

Functoriality of predicates P is natural in $b : C$, dinatural in $a : C$:

$$\dfrac{\dfrac{}{[z : C]\ p : P(z) \vdash p : P(z)}\ (\mathsf{var})}{[a : C^{\mathsf{op}}, b : C]\ e : \hom(a, b), p : P(\overline{a}) \vdash J(p) : P(b)}\ (J)$$

Directed type theory with dinaturality – non-examples

Failure of symmetry for directed equality

The restrictions do *not* allow us to obtain directed equality is symmetric:

$$[a : \mathbb{C}^{\mathsf{op}}, b : \mathbb{C}] \; e : \hom(a, b) \nvdash \mathsf{sym} : \hom(\bar{b}, \overline{a})$$

$\hom(a, b)$ cannot be contracted: a, b must appear *positively* in conclusion.

- Semantically, the interval $I := \{0 \to 1\}$ is a counterexample to derivability of this entailment in the syntax.

Directed type theory: equational theory

- A judgement $\boxed{[\Gamma]\ \Phi \vdash \alpha = \beta : P}$ for equality of entailments (in **Set**).
- The computation rule for J is expressed using equality of entailments:

$$\frac{}{[z : C, \Gamma]\ \Phi \vdash J(h)[\mathsf{refl}_z] = h : P}\ (J\text{-comp})$$

where we used cut of dinaturals (with refl), *which for J always works!*

Example (Left unitality for composition)

$$\frac{}{[z : C, c : C]\ g : \hom(\overline{z}, c) \vdash \mathsf{comp}[\mathsf{refl}_z, g] = g : \hom(\overline{z}, c)}\ (J\text{-comp})$$

Example (Terms send identities to identities)

$$\frac{}{[z : C]\ \Phi \vdash \mathsf{map}[\mathsf{refl}_z] = F^*(\mathsf{refl}_z) : \hom(F(\overline{z}), F(z))}\ (J\text{-comp})$$

Dependent directed J

- *What if we want to prove unitality on the right, or associativity?*
- There is a "dependent version of J" for equality of entailments:

$$\frac{[z : C, \Gamma]\ \Phi(z, \overline{z}) \vdash \alpha[\mathsf{refl}_z] = \beta[\mathsf{refl}_z] : P(\overline{z}, z)}{[a : C^{\mathsf{op}}, b : C, \Gamma]\ e : \hom_C(a, b), \Phi(\overline{a}, \overline{b}) \vdash \alpha[e] = \beta[e] : P(a, b)}\ (J\text{-eq})$$

- *Intuition: two dinaturals α, β are equal everywhere if they agree on* refl*.*
- **Semantics:** crucially, using dinaturality!

Example (Unitality on the right, associativity)

$$\frac{\dfrac{}{[w : C]\ \cdot \vdash \mathsf{refl}_w \mathbin{;} \mathsf{refl}_w = \mathsf{refl}_w : \hom(\overline{w}, w)}\ (J\text{-comp})}{[a : C^{\mathsf{op}}, z : C]\ f : \hom(a, z) \vdash f \mathbin{;} \mathsf{refl}_z = f : \hom(a, z)}\ (J\text{-eq})$$

To prove associativity, simply contract $f : \hom(a, b)$:

$$\frac{\dfrac{}{[z, c, d : C]\quad g : \hom(\overline{z}, c), h : \hom(\overline{c}, d) \vdash \mathsf{refl}_z \mathbin{;} (g \mathbin{;} h) = (\mathsf{refl}_z \mathbin{;} g) \mathbin{;} h : \hom(\overline{z}, d)}\ (J\text{-comp})}{[a, b, c, d : C]\ f : \hom(\overline{a}, b), g : \hom(\overline{b}, c), h : \hom(\overline{c}, d) \vdash f \mathbin{;} (g \mathbin{;} h) = (f \mathbin{;} g) \mathbin{;} h : \hom(\overline{a}, d)}\ (J\text{-eq})$$

Naturality for free

Example (Naturality of entailments)

Given a natural entailment α from P to Q,

$$\frac{}{[x : C]\ p : P(x) \vdash \alpha[p] : Q(x)}$$

we prove naturality, simply by contracting $f : \hom(a, b)$:

$$\dfrac{\dfrac{\dfrac{}{[z : C] \quad p : P(z) \vdash \alpha[p] = \alpha[p] : Q(z)}\ (\text{=-refl})}{[z : C] \quad p : P(z) \vdash \mathsf{transp}_Q[\mathsf{refl}, \alpha[p]] = \alpha[\mathsf{transp}_P[\mathsf{refl}, p]] : Q(z)}\ (J\text{-comp})}{[a : C^{\mathsf{op}}, b : C]\ f : \hom(a, b), p : P(\overline{a}) \vdash \mathsf{transp}_Q[f, \alpha[p]] = \alpha[\mathsf{transp}_P[f, p]] : Q(b)}\ (J\text{-eq})$$

- This also works for dinaturality because *transport is a natural.*

Naturality for free

Example (Natural transformations for terms)

Given a natural transformation α from F to G,

$$\frac{}{[x : C]\ \cdot \vdash \alpha : \hom_D(F(\overline{x}), G(x))}$$

We prove naturality of families simply by contracting $f : \hom(a, b)$:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{[z : C]\ \cdot \vdash \alpha = \alpha : \hom(F(\overline{z}), G(z))}\ (=\text{-refl})}{[z : C]\ \cdot \vdash \mathsf{refl}_{F(z)} \mathbin{;} \alpha = \alpha \mathbin{;} \mathsf{refl}_{G(z)} : \hom(F(\overline{z}), G(z))}\ (J\text{-comp})}{[z : C]\ \cdot \vdash \mathsf{map}_F[\mathsf{refl}_z] \mathbin{;} \alpha = \alpha \mathbin{;} \mathsf{map}_G[\mathsf{refl}_z] : \hom(F(\overline{z}), G(z))}\ (J\text{-comp})}{[a : C^{\mathsf{op}}, b : C]\ f : \hom(a, b) \vdash \mathsf{map}_F[f] \mathbin{;} \alpha = \alpha \mathbin{;} \mathsf{map}_G[f] : \hom(F(a), G(b))}\ (J\text{-eq})$$

- We can internalize all these transformations using ends:

$$[\,]\ \cdot \vdash \alpha : \mathsf{Nat}(F, G) := \int_{x:C} \hom_D(F(\overline{x}), G(x))$$

$$[\,]\ \cdot \vdash \alpha : \mathsf{Nat}(P, Q) := \int_{x:C} P(\overline{x}) \Rightarrow Q(x)$$

Directed type theory: logical rules

- Logical rules are given as isomorphisms in "adjoint form":

$$\frac{[\Gamma]\ \Phi \vdash P \times Q}{[\Gamma]\ \Phi \vdash P, \qquad [\Gamma]\ \Phi \vdash Q}\ \text{(prod)}$$

- Dinaturals can be curried: intuitively, all positions invert polarity:

$$\frac{[x : \Gamma]\ A(\overline{x}, x), \Phi(\overline{x}, x) \vdash B(\overline{x}, x)}{[x : \Gamma] \qquad \Phi(\overline{x}, x) \vdash A(x, \overline{x}) \Rightarrow B(\overline{x}, x)}\ \text{(exp)}$$

- Rules for (co)ends in "adjoint" form:

$$\frac{[a : C, \Gamma]\ \Phi \vdash Q(\overline{a}, a)}{[\Gamma]\ \Phi \vdash \int_{a:C} Q(\overline{a}, a)}\ \text{(end)} \qquad \frac{[\Gamma]\ \left(\int^{a:C} Q(\overline{a}, a)\right), \Phi \vdash P}{[a : C, \Gamma]\ Q(\overline{a}, a), \Phi \vdash P}\ \text{(coend)}$$

- This is the presentation $\forall/\exists$-as-adjoints, up to composition of dinaturals.

Semantics of directed J

- This semantic result is where the restrictions of J come from:

Theorem

*There is a bijection (natural in $P, Q : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to$ **Set**) between sets of dinaturals and sets of **naturals** like this:*

$$\frac{P \xrightarrow{\cdot\cdot} Q}{\hom(a,b) \longrightarrow P^{\mathsf{op}}(b,a) \Rightarrow Q(a,b)}$$

Proof. *precisely by Yoneda: pick the identities, use (di)naturality.*

- This is where J comes from:

$$\left.\frac{\dfrac{[z : C, \Gamma] \qquad \Phi(\overline{z}, z) \vdash P(\overline{z}, z)}{[a : C^{\mathsf{op}}, b : C, \Gamma] \;\; \hom_C(a,b) \vdash \Phi(b,a) \Rightarrow P(a,b)}}{[a : C^{\mathsf{op}}, b : C, \Gamma] \;\; \hom_C(a,b), \Phi(\overline{b}, \overline{a}) \vdash P(a,b)}\,(\mathsf{exp})\right\}(J)$$

- **Syntax:** all rules for $\hom$ are derivable $\iff$ (J) is an iso is derivable.

(Co)end calculus

- Using our rules we can prove category theory theorems "logically".
- We use (co)end calculus-style reasoning, i.e., we show that two presheaves are isomorphic using Yoneda.
- Adjoint form is better suited to (co)end calculus style reasoning: term-based reasoning is hard because of dinaturality.
- Rules for (co)ends as quantifiers + directed equality:
  - (Co)Yoneda,
  - Adjointess of Kan extensions via (co)ends,
  - Presheaves are closed under exponentials,
  - Associativity of composition of profunctors,
  - Right lifts in profunctors,
  - (Co)ends preserve limits,
  - Adjointness of (co)ends in natural transformations,
  - Characterization of dinaturals as certain ends,
  - Frobenius property of (co)ends using exponentials.

(Co)end calculus with dinaturality (1)

Yoneda lemma: $(\llbracket P \rrbracket, \llbracket \Gamma \rrbracket : \llbracket C \rrbracket \to \mathbf{Set})$

$$\dfrac{\dfrac{[a : C]\ \Gamma(a) \vdash \displaystyle\int_{x:C} \hom_C(a, \overline{x}) \Rightarrow P(x)}{[a : C, x : C]\ \Gamma(a) \vdash \hom_C(a, \overline{x}) \Rightarrow P(x)}\ \text{(end)}}{\dfrac{[a : C, x : C]\ \ \hom_C(\overline{a}, x) \times \Gamma(a) \vdash P(x)}{[z : C]\ \Gamma(z) \vdash P(z)}\ (\hom)}\ \text{(exp)}$$

CoYoneda lemma:

$$\dfrac{[a : C]\ \displaystyle\int^{x:C} \hom_C(\overline{x}, a) \times P(x) \vdash \Gamma(a)}{\dfrac{[a : C, x : C]\ \ \hom_C(\overline{a}, x) \times P(a) \vdash \Gamma(x)}{[z : C]\ P(z) \vdash \Gamma(z)}\ (\hom)}\ \text{(coend)}$$

(Co)end calculus with dinaturality (2)

Presheaves are cartesian closed: ($[\![\Gamma]\!], [\![A]\!], [\![B]\!] : [\![C]\!] \to$ **Set**)

$$
\dfrac{\dfrac{\dfrac{\dfrac{\begin{aligned}[x : C]\ \Gamma(x) \vdash (A \Rightarrow B)(x) \\ := \mathsf{Nat}(\hom_C(x, -) \times A, B) \\ \cong \int_{y:C} \hom_C(x, \overline{y}) \times A(\overline{y}) \Rightarrow B(y)\end{aligned}}{[x : C, y : C]\ \Gamma(x) \vdash \hom_C(x, \overline{y}) \times A(\overline{y}) \Rightarrow B(y)}\ \text{(end)}}{[x : C, y : C]\ A(y) \times \hom_C(\overline{x}, y) \times \Gamma(x) \vdash B(y)}\ \text{(exp)}}{[y : C]\ A(y) \times \left(\int^{x:C} \hom_C(\overline{x}, y) \times \Gamma(x)\right) \vdash B(y)}\ \text{(coend)}}{[y : C]\ A(y) \times \Gamma(y) \vdash B(y)}\ \text{(coYoneda)}
$$

(Co)end calculus with dinaturality (3)

Right Kan extensions via ends are right adjoints to precomposition with $F : C \to D$ ($P : C \to \mathbf{Set}, \mathit{\Gamma} : D \to \mathbf{Set}$):

$$
\cfrac{\cfrac{\begin{array}{r}[y : D]\ \mathit{\Gamma}(y) \vdash (\mathsf{Ran}_F P)(y) \\ := \displaystyle\int_{x:C} \hom_D(y, F(\overline{x})) \Rightarrow P(x)\end{array}}{[x : C, y : D]\ \mathit{\Gamma}(y) \vdash \hom_D(y, F(\overline{x})) \Rightarrow P(x)}\ \text{(end)}}{\cfrac{[x : C, y : D]\ \hom_D(\overline{y}, F(x)) \times \mathit{\Gamma}(y) \vdash P(x)}{\cfrac{[x : C]\ \displaystyle\int^{y:D} \hom_D(\overline{y}, F(x)) \times \mathit{\Gamma}(y) \vdash P(x)}{[x : C]\ \mathit{\Gamma}(F(x)) \vdash P(x)}\ \text{(coYoneda)}}\ \text{(coend)}}\ \text{(exp)}
$$

(Co)end calculus with dinaturality (4)

Fubini for ends (Γ : [] prop, $P : [C, D]$ prop)

$$
\dfrac{\dfrac{\dfrac{\dfrac{[\,]\ \Gamma \vdash \int_{x:C} \int_{y:D} P(\overline{x}, x, \overline{y}, y)}{[x : C]\ \Gamma \vdash \int_{y:D} P(\overline{x}, x, \overline{y}, y)}\ \text{(end)}}{[x : C, y : D]\ \Gamma \vdash P(\overline{x}, x, \overline{y}, y)}\ \text{(end)}}{[y : D, x : C]\ \Gamma \vdash P(\overline{x}, x, \overline{y}, y)}\ \text{(structural property)}
$$

$$
\dfrac{\dfrac{[y : D, x : C]\ \Gamma \vdash P(\overline{x}, x, \overline{y}, y)}{[y : D]\ \Gamma \vdash \int_{x:C} P(\overline{x}, x, \overline{y}, y)}\ \text{(end)}}{[\,]\ \Gamma \vdash \int_{y:D} \int_{x:C} P(\overline{x}, x, \overline{y}, y)}\ \text{(end)}
$$

Conclusion and future work

*We have seen how dinaturality allows us to give a semantic interpretation to a first-order directed type theory in **Cat** with quantifiers, where directed equality is given by* $\hom$*-functors and quantifiers by (co)ends.*

Future work:

1. Big piece missing from the story: compositionality of dinaturals.
   - *Claim:* non-compositionality is intrinsic to **Cat**, like failure of UIP.
   - Find suitable structures axiomatizing composition of dinaturality (e.g., operads/multicategories but with explicit variances of variables.).
2. Long-term future: now that types are categories,
   - Internalize semantics of type theory inside type theory (e.g., dQIIT).
   - Revisit category-theoretic concepts logically.
3. Immediate future: a working notion of *dinatural context extension* $\rightsquigarrow$ towards *dependent dinatural directed type theory*.

The $\int$.

Paper: *"Directed equality with dinaturality"* (`arXiv:2409.10237`)
Website: `iwilare.com` (← updated version is here!)

Thank you for the attention!