Di- is for Directed:
First-Order Directed Type Theory via Dinaturality

Andrea Laretto, Fosco Loregian, Niccolò Veltri

Tallinn University of Technology

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Motivation

Category theory is *hard*.

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The claim of this talk: category theory $=$ logic.

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This is the Yoneda lemma!

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We want to prove things like the Yoneda lemma just as easily as the equivalence above.

Proof of Yoneda in dinatural directed type theory

The previous equivalence in first-order logic:

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Our **formal** proof for the Yoneda lemma $\mathsf{Nat}(\hom_C(a,-),P) \cong P(a)$:

$$\dfrac{\dfrac{\dfrac{[a:C]\ \Phi \vdash \displaystyle\int_{x:C} \hom_C(a,\overline{x}) \Rightarrow P(x)}{[a:C, x:C]\ \Phi \vdash \hom_C(a,\overline{x}) \Rightarrow P(x)}\ (\textstyle\int)}{[a:C, x:C]\ \ \hom_C(\overline{a},x) \times \Phi \vdash P(x)}\ (\Rightarrow)}{[a:C]\ \Phi \vdash P(a)}\ (\hom)$$

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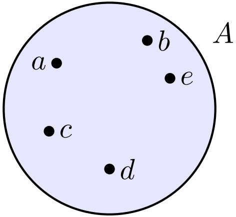

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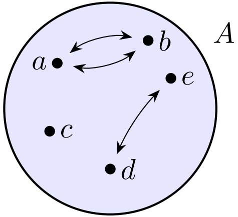

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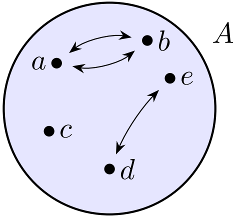

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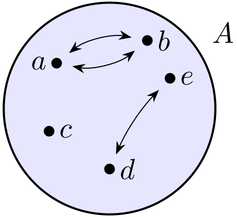

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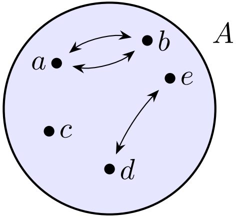

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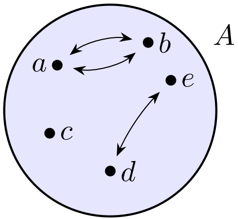

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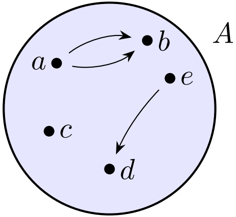

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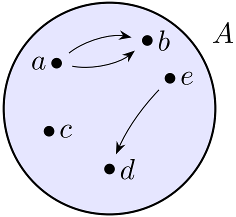

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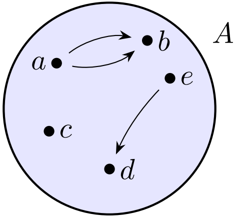

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$\to$ Type theory as a unifying framework for rewriting, processes, transitions, etc.

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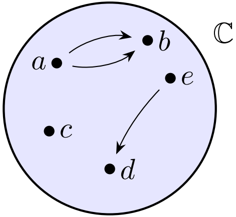

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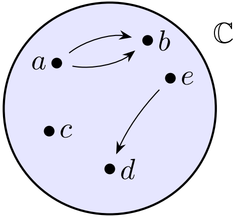

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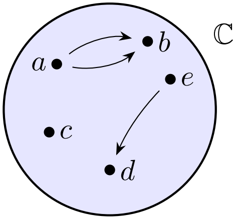

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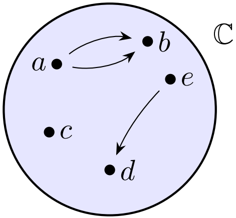

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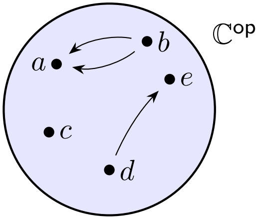

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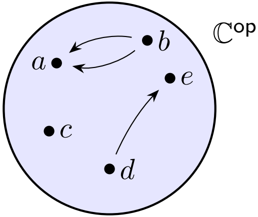

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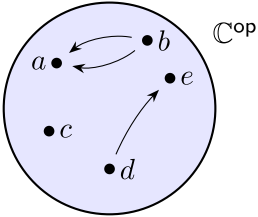

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$\forall (x : C).\; f(x) =_D g(x)$ $\int_{x:\mathbb{C}} \hom_{\mathbb{D}}(F(x), G(x))$	Pointwise equality of functions Natural transformations

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$\rightarrow$ a **simple** description of directed type theory,

$\rightarrow$ simple *logical* proofs of theorems in category theory.

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We present a first-order (i.e., non-dependent) directed type theory (i.e. proof-relevant), and solve these two problems using **dinaturality**.

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Equality predicates	$\hom : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathsf{Set}$
Entailments	Dinatural transformations (not required to compose)
Quantifiers $\forall, \exists$	Ends $\int_{x:\mathbb{C}} P(\overline{x}, x)$, coends $\int^{x:\mathbb{C}} P(\overline{x}, x)$.

Syntax – simple types and terms

- Judgement $\boxed{C \text{ type}}$ for types:

$$\frac{C \text{ type}}{C^{\mathsf{op}} \text{ type}} \qquad \frac{C \text{ type} \qquad D \text{ type}}{C \times D \text{ type}} \qquad \frac{C \text{ type} \qquad D \text{ type}}{[C, D] \text{ type}} \qquad \frac{}{\top \text{ type}}$$

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Notation: *if* $x : C$ *in* Γ, *then* $\overline{x} : C^{\mathsf{op}}$ *in* Γ^{op}.

Syntax – polarity and variance

- I can use variables *"incorrectly"*, regardless of the outermost op: $x : C$, $\overline{x} : C^{\mathsf{op}}$:

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$\rightsquigarrow$ *dinatural transformations*!

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- Directed equality elimination:

$$\dfrac{[z : C, \Gamma] \qquad \Phi(z, \overline{z}) \vdash h : P(\overline{z}, z)}{[a : C^{\mathsf{op}}, b : C, \Gamma]\ e : \hom_C(a, b), \Phi(\overline{a}, \overline{b}) \vdash J(h) : P(a, b)}\ (J)$$

If I have a directed equality $e : \hom_C(a, b)$ in context,

- I can contract it only if a, b appear *correctly* in the conclusion P,
- and a, b appear *incorrectly* in the context Φ.

- Then, it is enough to prove that P holds "on the diagonal" $z : C$.

Dinatural directed type theory – examples

Example (Transitivity of directed equality)

$$\frac{}{[a : C^{\mathsf{op}}, b : C, c : C]\ f : \hom(a, b),\ g : \hom(\bar{b}, c) \vdash {\color{blue}?} \qquad \hom(a, c)}$$

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Rule (J) can be applied: a, b appear correctly in conclusion ($\bar{b}$ does not)

and incorrectly in context (a does not).

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$$\dfrac{\dfrac{}{[z : C, c : C] \qquad g : \hom(\overline{z}, c) \vdash g : \hom(\overline{z}, c)}\ (\mathsf{id})}{[a : C^{\mathsf{op}}, b : C, c : C]\ f : \hom(a, b),\ g : \hom(\overline{b}, c) \vdash J(g) : \hom(a, c)}\ (J)$$

We contract $f : \hom(a, b)$.
Rule (J) can be applied: a, b appear correctly in conclusion ($\overline{b}$ does not)
and incorrectly in context (a does not).

Dinatural directed type theory – examples

Example (Congruence / terms are functors)

Given a term $C \vdash F : D$:

$$\dfrac{\dfrac{}{[z : D] \;\vdash\; \mathsf{refl}_x : \hom_D(\overline{x}, x)}\,(\mathsf{refl})}{\dfrac{[z : C] \;\vdash\; \mathsf{refl}_{F(x)} : \hom_D(F(\overline{z}), F(z))}{[a : C^{\mathsf{op}}, b : C]\; e : \hom_C(a,b) \vdash J(\mathsf{refl}_{F(x)}) : \hom_D(F(a), F(b))}\,(J)}\,(\mathsf{reidx})$$

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$$

Example (Transport / predicates are functors)

Given a predicate $[x : C]\; P(x)$ prop:

$$
\cfrac{\cfrac{}{[z : C]\; p : P(z) \vdash p : P(z)}\;(\mathsf{id})}{[a : C^{\mathsf{op}}, b : C]\; e : \hom(a, b), p : P(\overline{a}) \vdash J(p) : P(b)}\;(J)
$$

Dinatural directed type theory – non-examples

Failure of symmetry for directed equality

The restrictions do *not* allow us to obtain directed equality is symmetric:

$$[a : C^{\mathsf{op}}, b : C]\ e : \hom(a, b) \nvdash \mathsf{sym} : \hom(\bar{b}, \overline{a})$$

$\hom(a, b)$ cannot be contracted: a, b must appear *correctly* in conclusion.

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- By soundness, the interval $I := \{0 \to 1\}$ is a counterexample to derivability in the syntax.

Directed type theory: equational theory

- A judgement $\boxed{[\Gamma]\ \Phi \vdash \alpha = \beta : P}$ for equality of entailments.

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Example (Left unitality for composition)

Recall that $\mathsf{compose}[f, g] := J(g)[f, g]$:

$$\frac{}{[z : C, c : C]\ g : \hom(\overline{z}, c) \vdash \mathsf{compose}[\mathsf{refl}_z, g] = g : \hom(\overline{z}, c)}\ (J\text{-comp})$$

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Example (Functors send identities to identities)

$$\frac{}{[z : C]\ \ \vdash \mathsf{map}_F[\mathsf{refl}_z] = \mathsf{refl}_{F(z)} : \hom(F(\overline{z}), F(z))}\ (J\text{-comp})$$

Directed equality induction

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Example (Unitality on the right)

$$\dfrac{\dfrac{}{[w : C]\ \ \vdash \mathsf{refl}_w \mathbin{;} \mathsf{refl}_w = \mathsf{refl}_w : \hom(\overline{w}, w)}\ (J\text{-comp})}{[a : C^{\mathsf{op}}, z : C]\ f : \hom(a, z) \vdash f \mathbin{;} \mathsf{refl}_z = f : \hom(a, z)}\ (J\text{-eq})$$

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Example (Associativity)

$$\dfrac{\dfrac{\dfrac{}{[z, c, d : C]\ g : \hom(\overline{z}, c), h : \hom(\overline{c}, d) \vdash g ; h = g ; h : \hom(\overline{z}, d)}\ (=\text{-refl})}{[z, c, d : C]\ g : \hom(\overline{z}, c), h : \hom(\overline{c}, d) \vdash \mathsf{refl}_z ; (g ; h) = (\mathsf{refl}_z ; g) ; h : \hom(\overline{z}, d)}\ (J\text{-comp})}{[a, b, c, d : C]\ f : \hom(\overline{a}, b), g : \hom(\overline{b}, c), h : \hom(\overline{c}, d) \vdash f ; (g ; h) = (f ; g) ; h : \hom(\overline{a}, d)}\ (J\text{-eq})$$

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Example (Functoriality)

$$\cfrac{\cfrac{\overline{[z, c : C]\quad g : \hom(\overline{z}, c) \vdash \mathsf{map}_F[g] = \mathsf{map}_F[g] : \hom(F(\overline{z}), F(c))}\ (=\text{-refl})}{[z, c : C]\quad g : \hom(\overline{z}, c) \vdash \mathsf{map}_F[\mathsf{refl}_z \,;\, g] = \mathsf{refl}_{F(z)} \,;\, \mathsf{map}_F[g] : \hom(F(\overline{z}), F(c))}\ (J\text{-comp})}{[a, b, c : C]\ f : \hom(\overline{a}, b),\ g : \hom(\overline{b}, c) \vdash \mathsf{map}_F[f \,;\, g] = \mathsf{map}_F[f] \,;\, \mathsf{map}_F[g] : \hom(F(\overline{a}), F(c))}\ (J\text{-eq})$$

Naturality for free!

Example (Naturality for terms)

Given a natural transformation α from F to G,

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Given a natural transformation α from F to G,

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we prove naturality by contracting $f : \hom(a, b)$:

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Given a natural entailment α from P to Q,

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Directed type theory: logical rules

- Logical rules are given in "adjoint form", i.e., as bijections:

$$\frac{[\Gamma]\ \Phi \vdash P \times Q}{[\Gamma]\ \Phi \vdash P, \qquad [\Gamma]\ \Phi \vdash Q}\ (\text{prod})$$

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- Dinaturals can be curried (all positions invert polarity):

$$\frac{[x : \Gamma]\ A(\overline{x}, x), \Phi(\overline{x}, x) \vdash B(\overline{x}, x)}{[x : \Gamma] \qquad \Phi(\overline{x}, x) \vdash A(x, \overline{x}) \Rightarrow B(\overline{x}, x)} \ (\Rightarrow)$$

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- Rules for (co)ends as "adjoints":

$$\frac{[a : C, \Gamma]\ \Phi \vdash Q(\overline{a}, a)}{[\Gamma]\ \Phi \vdash \int_{a:C} Q(\overline{a}, a)}\ (\textstyle\int) \qquad \frac{[\Gamma]\ \left(\int^{a:C} Q(\overline{a}, a)\right), \Phi \vdash P}{[a : C, \Gamma]\ Q(\overline{a}, a), \Phi \vdash P}\ (\mathsf{co}\textstyle\int)$$

(Co)end calculus

- We can prove theorems in category theory *logically*.

(Co)end calculus

- We can prove theorems in category theory *logically*.
- Rules for (co)ends as quantifiers + directed equality:
  1. (Co)Yoneda,
  2. Adjointess of Kan extensions via (co)ends,
  3. Presheaves are closed under exponentials,
  4. Associativity of composition of profunctors,
  5. Right lifts in profunctors,
  6. (Co)ends preserve limits,
  7. Adjointness of (co)ends in natural transformations,
  8. Characterization of (di)naturals as ends,
  9. Frobenius property of (co)ends using exponentials,
  10. Contractibility of singletons: $\lim_{x} \operatorname{colim}_{y} \hom(x, y) \cong 1$.

(Co)end calculus with dinaturality (1)

Yoneda lemma: $(\llbracket P \rrbracket, \llbracket \Phi \rrbracket : C \to \mathsf{Set})$

$$\dfrac{\dfrac{\dfrac{[a : C]\ \Phi(a) \vdash \displaystyle\int_{x:C} \hom_C(a,\overline{x}) \Rightarrow P(x)}{[a : C, x : C]\ \Phi(a) \vdash \hom_C(a,\overline{x}) \Rightarrow P(x)}\ (\textstyle\int)}{[a : C, x : C]\ \hom_C(\overline{a},x) \times \Phi(a) \vdash P(x)}\ (\Rightarrow)}{[z : C]\ \Phi(z) \vdash P(z)}\ (J)$$

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$$
\begin{array}{c}
[a : C]\ \Phi(a) \vdash \displaystyle\int_{x:C} \hom_C(a, \overline{x}) \Rightarrow P(x) \\
\hline\hline
[a : C, x : C]\ \Phi(a) \vdash \hom_C(a, \overline{x}) \Rightarrow P(x)
\end{array}
\ (\textstyle\int)
$$

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\hline\hline
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$$

$$
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\end{array}
\ (J)
$$

CoYoneda lemma:

$$
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\hline\hline
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\end{array}
\ (\mathsf{co}\textstyle\int)
$$

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\hline\hline
[z : C]\ P(z) \vdash \Phi(z)
\end{array}
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$$

(Co)end calculus with dinaturality (2)

Presheaves are cartesian closed: $(\llbracket\Phi\rrbracket, \llbracket A\rrbracket, \llbracket B\rrbracket : C \to \mathsf{Set})$

$$\begin{aligned}
[x : C]\ \Phi(x) \vdash (A \Rightarrow B)(x) \\
:=\ & \mathsf{Nat}(\hom_C(x, -) \times A, B) \\
\cong & \int_{y:C} \hom_C(x, \overline{y}) \times A(\overline{y}) \Rightarrow B(y)
\end{aligned}$$

$$\frac{\dfrac{\dfrac{}{[x : C, y : C] \quad \Phi(x) \vdash \hom_C(x, \overline{y}) \times A(\overline{y}) \Rightarrow B(y)}\,(\textstyle\int)}{[x : C, y : C]\ A(y) \times \hom_C(\overline{x}, y) \times \Phi(x) \vdash B(y)}\,(\Rightarrow)}{[y : C] \quad A(y) \times \Phi(y) \vdash B(y)}\,(J)$$

(Co)end calculus with dinaturality (3)

Right Kan extensions are right adjoint to precomposing with $[\![F]\!] : C \to D$:

$$
\dfrac{\dfrac{\dfrac{[y : D]\ Q(y) \vdash (\mathsf{Ran}_F P)(y) := \displaystyle\int_{x:C} \hom_D(y, F(\overline{x})) \Rightarrow P(x)}{[x : C, y : D] \quad Q(y) \vdash \hom_D(y, F(\overline{x})) \Rightarrow P(x)}\ ({\textstyle\int})}{[x : C, y : D]\ \hom_D(\overline{y}, F(x)) \times Q(y) \vdash P(x)}\ (\Rightarrow)}{[x : C] \quad Q(F(x)) \vdash P(x)}\ (J)
$$

(Co)end calculus with dinaturality (4)

Fubini for ends ($[\,]\ \Phi$ propctx, $[C, D]\ P$ prop)

$$
\dfrac{\dfrac{\dfrac{\dfrac{[\,]\ \Phi \vdash \displaystyle\int_{x:C}\int_{y:D} P(\overline{x}, x, \overline{y}, y)}{[x : C]\ \Phi \vdash \displaystyle\int_{y:D} P(\overline{x}, x, \overline{y}, y)}\ (\textstyle\int)}{[x : C, y : D]\ \Phi \vdash P(\overline{x}, x, \overline{y}, y)}\ (\textstyle\int)}{[y : D, x : C]\ \Phi \vdash P(\overline{x}, x, \overline{y}, y)}\ \text{(structural property)}}{\dfrac{[y : D]\ \Phi \vdash \displaystyle\int_{x:C} P(\overline{x}, x, \overline{y}, y)}{[\,]\ \Phi \vdash \displaystyle\int_{y:D}\int_{x:C} P(\overline{x}, x, \overline{y}, y)}\ (\textstyle\int)}\ (\textstyle\int)
$$

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We have seen how dinaturality gives us a first-order directed type theory, which allows us to do category theory logically and in a simple way.

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   - Synthetic rewriting, synthetic models of type theory (e.g., dQIIT).
   - Revisit more category theory, logically.
3. Immediate future: we have a notion of *dinatural context extension* $\rightsquigarrow$ towards *dependent dinatural directed type theory*.

The $\int$.

Paper: *"Di- is for Directed: First-Order Directed Type Theory via Dinaturality"* (arXiv:2409.10237)
Website: iwilare.com

Thank you for the attention!

Where J comes from

Theorem

*There is a bijection (natural in $P, Q : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathsf{Set}$) between sets of dinaturals and sets of **naturals** like this:*

$$\frac{P \xrightarrow{\textit{dinat}} Q}{\mathrm{hom}(a,b) \longrightarrow P^{\mathsf{op}}(b,a) \Rightarrow Q(a,b)}$$

Proof. *precisely by Yoneda: pick the identities, use (di)naturality.*

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$$\left.\begin{array}{c} \dfrac{\dfrac{[z : C] \qquad \Phi(\overline{z}, z) \vdash P(\overline{z}, z)}{[a : C^{\mathsf{op}}, b : C] \;\; \hom_C(a,b) \vdash \Phi(b,a) \Rightarrow P(a,b)}}{[a : C^{\mathsf{op}}, b : C] \;\; \hom_C(a,b), \Phi(\overline{b}, \overline{a}) \vdash P(a,b)} \; (\Rightarrow) \end{array}\right\} (J)$$

- **Thm:** all rules for $\hom$ are derivable $\iff$ (J) is a bijection.

Homotopical interpretation of dinaturality

We have maps both ways:

$$\frac{[\,] \qquad \top \vdash P}{\overline{\overline{[x : C]\ x = x \vdash P}}}$$

but in MLTT they are not isomorphic.

In DTT, we do not even have both maps!

$$\frac{[\,] \qquad\qquad \top \vdash P}{\overline{\overline{[x : \mathbb{C}]\ \ \mathrm{hom}(\overline{x}, x) \vdash P}}}$$

We only have a map from top to bottom.