Categorical Semantics for Counterpart-based Temporal Logics in Agda

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Outline

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4. Conclusion and future work

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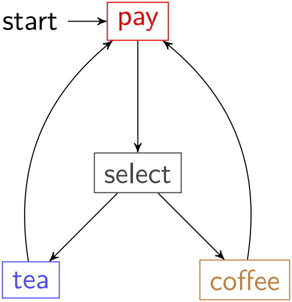

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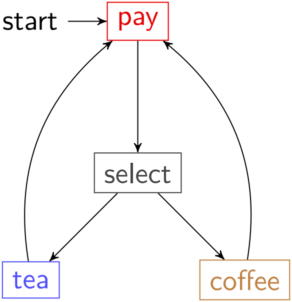

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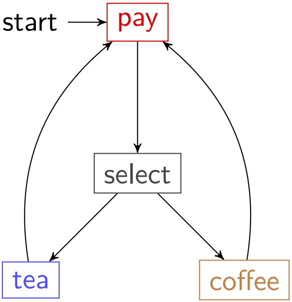

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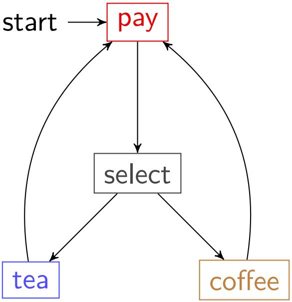

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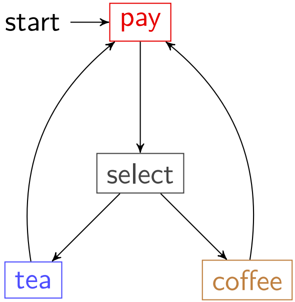

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3. Use a program to *check* that the *model* **satisfies** the formula

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- *Yes!* Using **counterpart models** and quantified temporal logics

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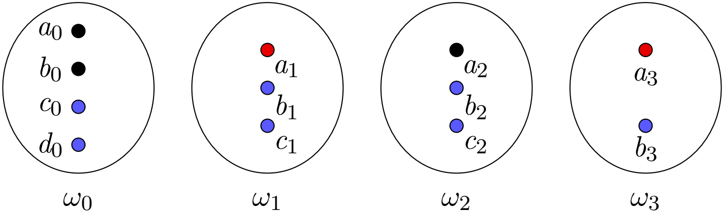

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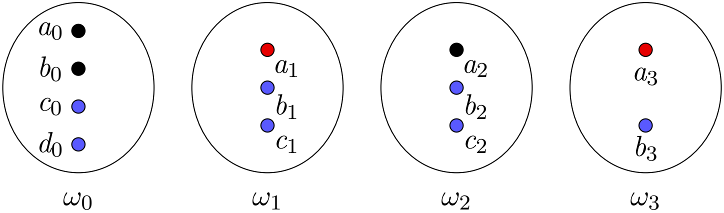

How do we represent transitions?

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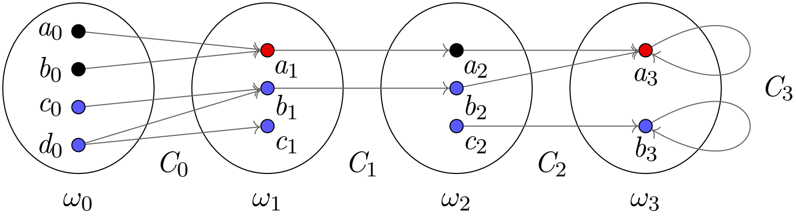

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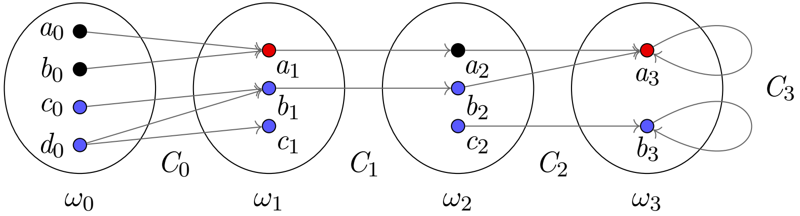

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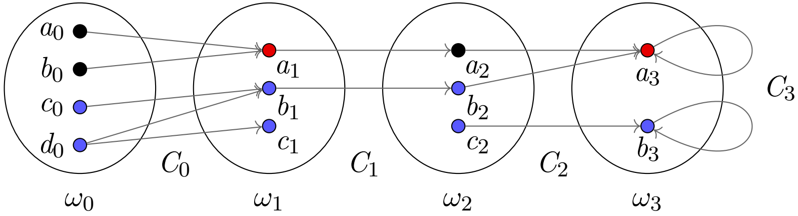

- Intuition: individuals connected by a relation *are the same after one step*
- We call these sequences of worlds and relations **counterpart traces**

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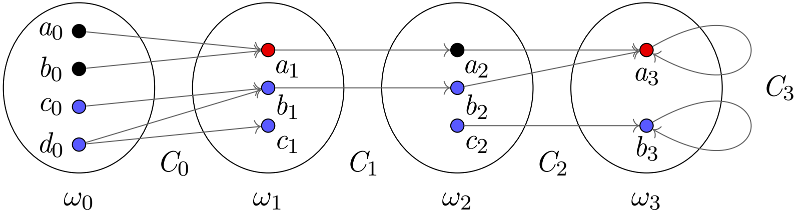

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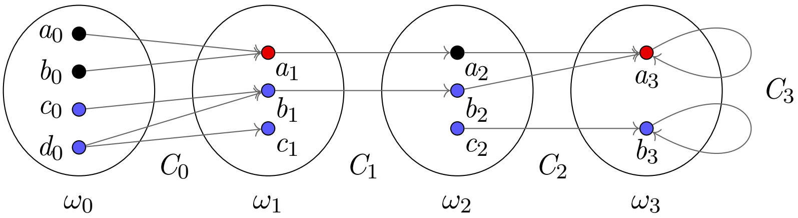

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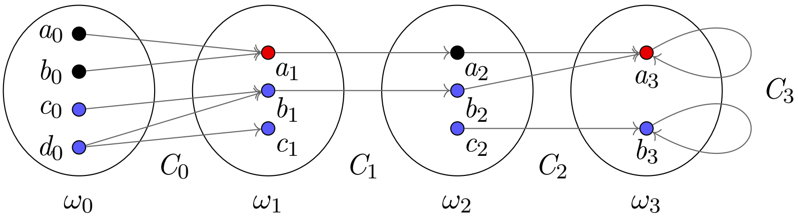

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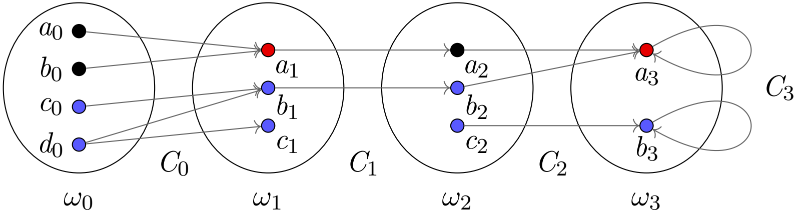

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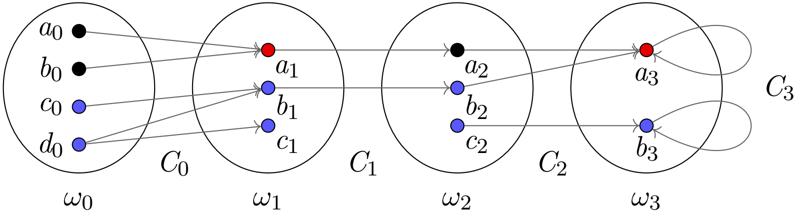

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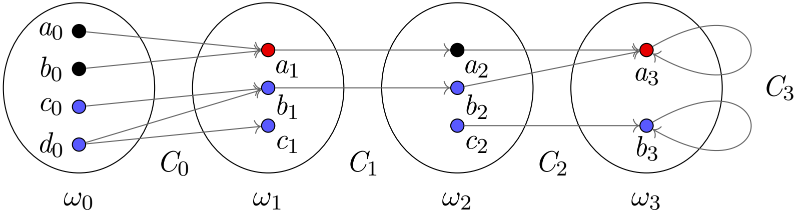

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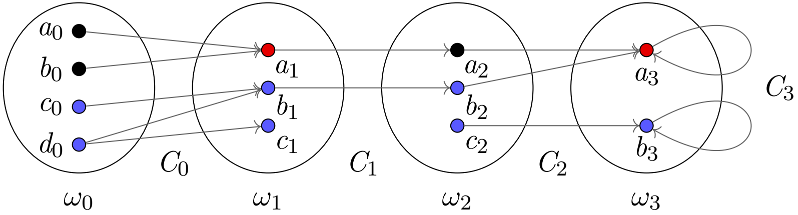

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$$
\begin{tikzcd}
\omega_0 \arrow[loop left, "\mathsf{id}_{\omega_0}"] \arrow[r, "f"] & \omega_1 \arrow[r, "g"] \arrow[rr, bend left, "g\,;\,h"] & \omega_2 \arrow[r, bend left, "h'"] \arrow[r, bend right, "h"'] & \omega_3 \arrow[loop right, "k"]
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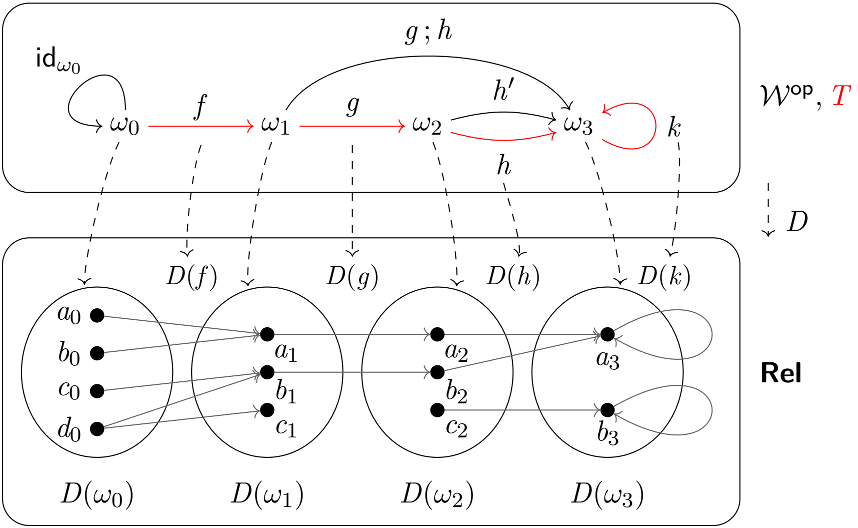
$\mathcal{W}^{\mathsf{op}}$, T

Relational presheaves – Example



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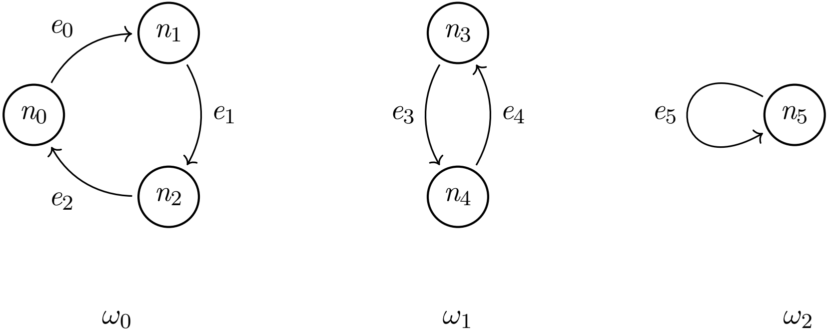

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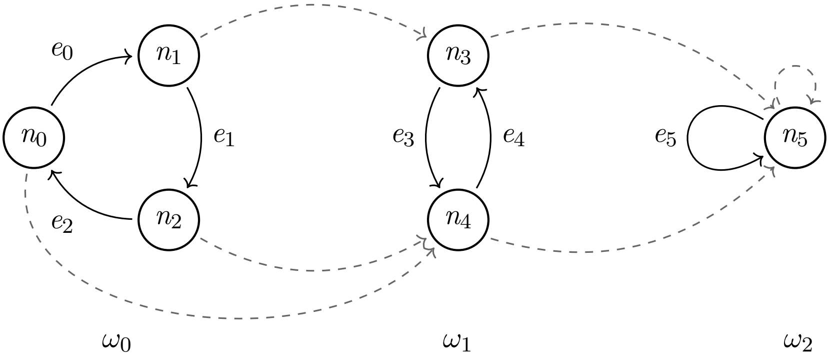

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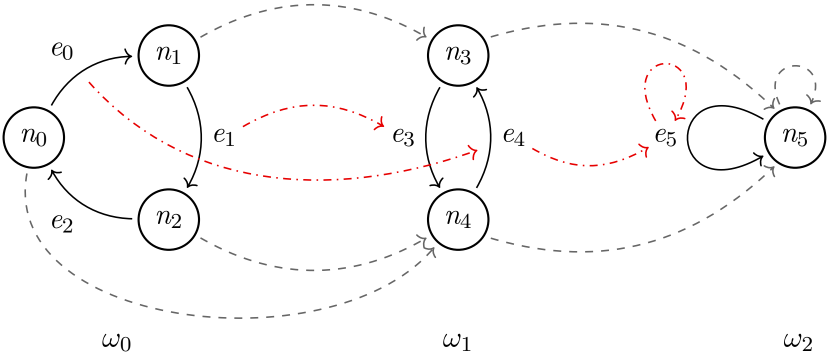

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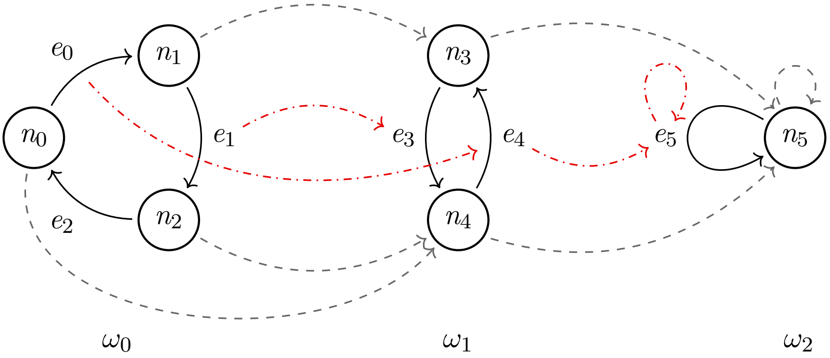

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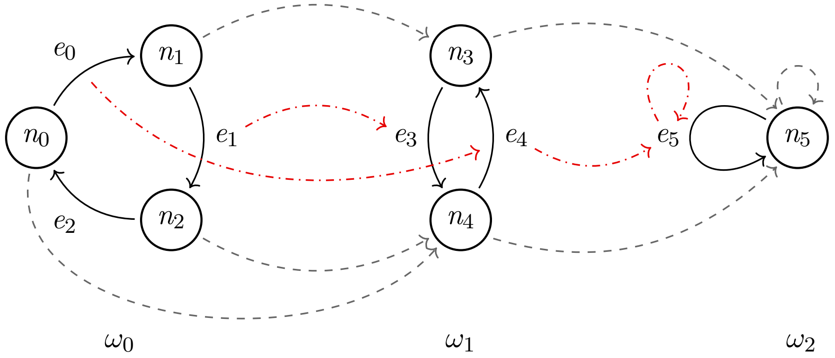

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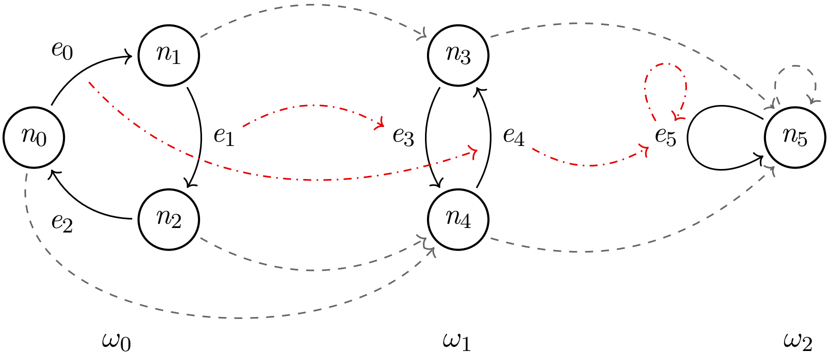

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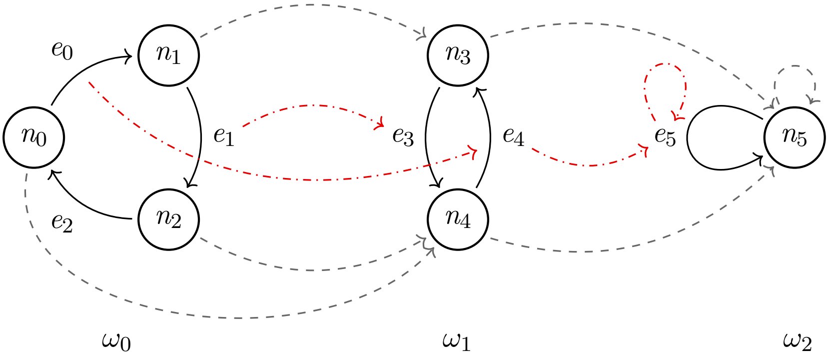

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Algebraic QLTL with relational presheaves

$$\begin{tikzcd} \omega_0 \arrow[r, "f_0"] & \omega_1 \arrow[r, "f_1"] & \omega_2 \arrow[loop right, "f_2"] \end{tikzcd}$$

$\mathcal{W}^{\mathsf{op}}$, T

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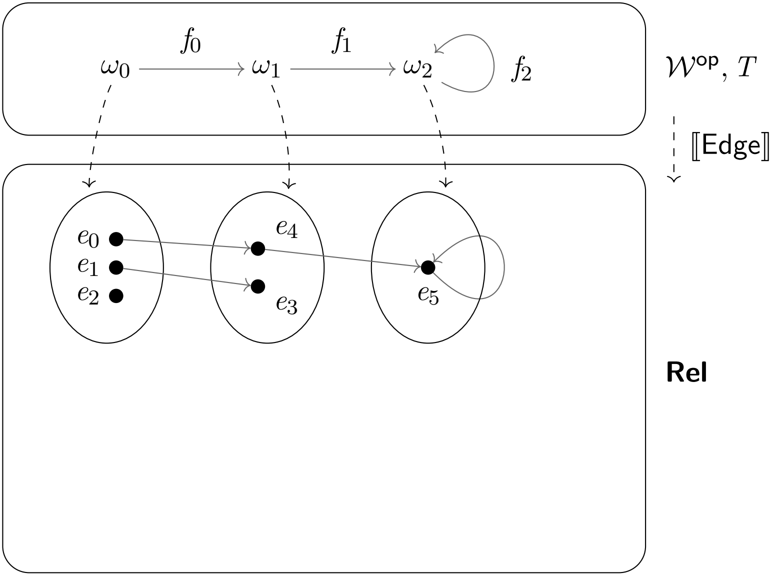

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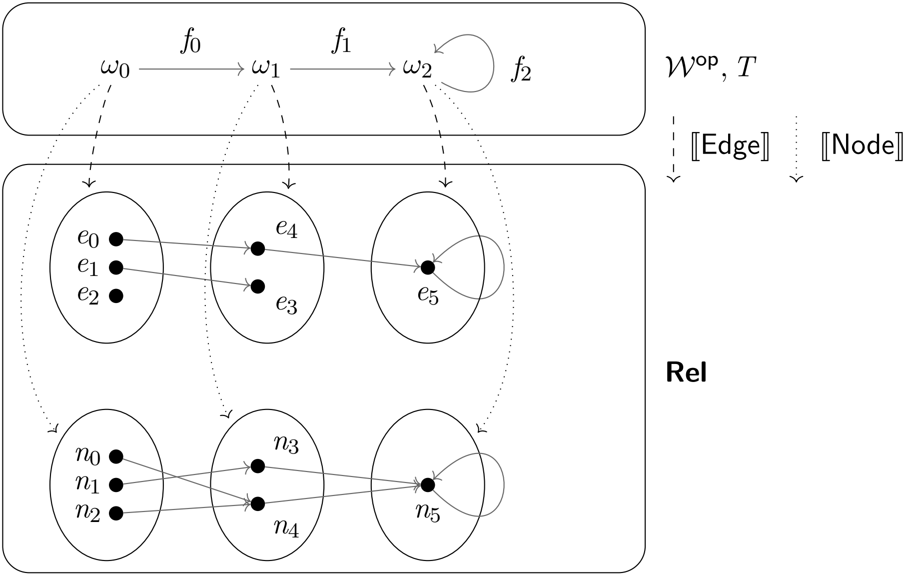

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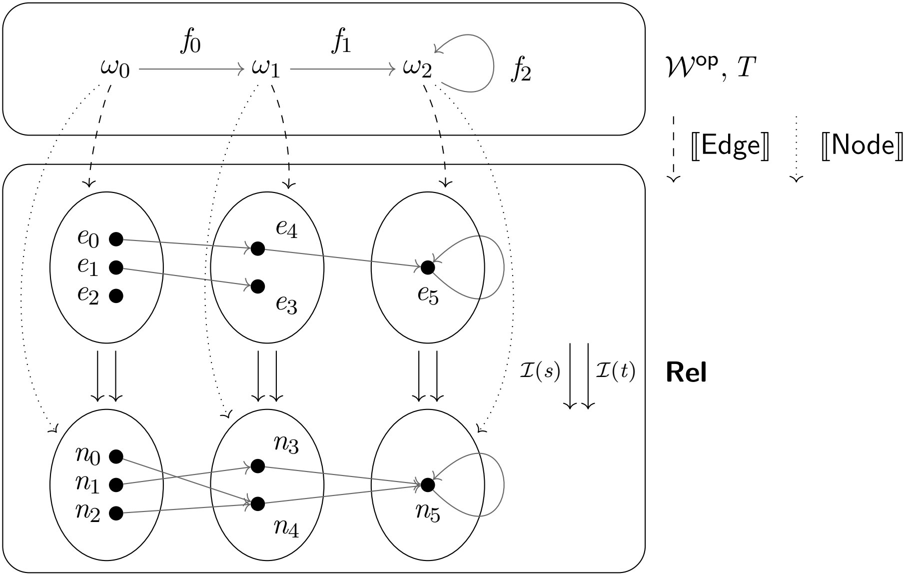

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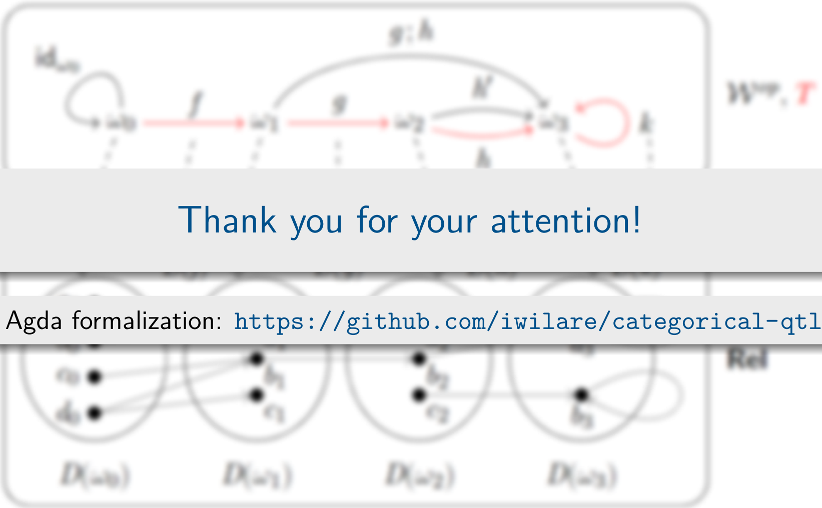
- Formalizing and using category theory in proof assistants: [Carette, Hu, 2021], [Gross, Chlipala, Spivak, 2014]



Thank you for your attention!

Agda formalization: `https://github.com/iwilare/categorical-qtl`

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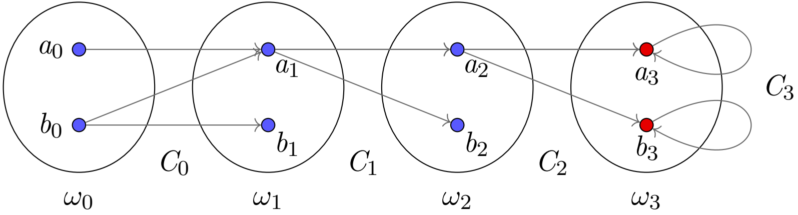

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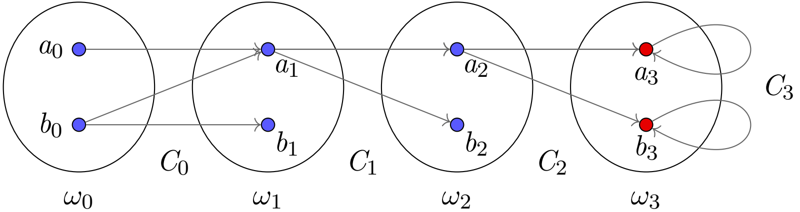

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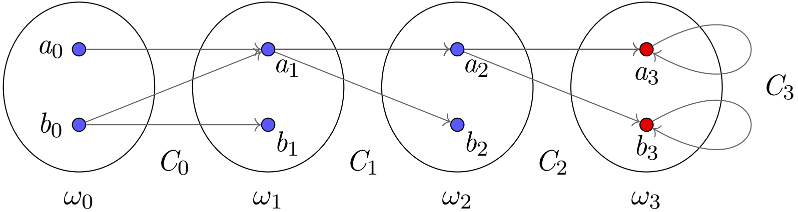

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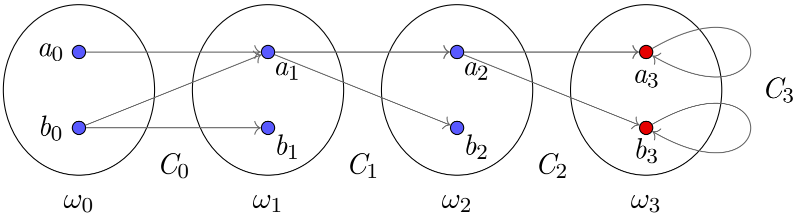

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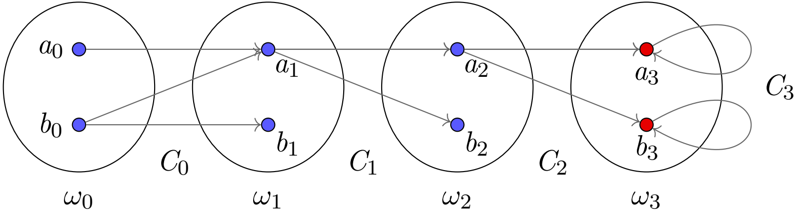

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