Completeness for Categories of Generalized Automata ((Co)algebraic pearls) / CALCO 2023

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Context

- Many ways of *categorifying* automata theory in category theory [Adámek-Trnková, 1990], [Rutten, 2000], [Jacobs, 2006]
- Very long tradition, with some works already from the 1970s [Ehrig et al. 1974], [Naudé, 1977], [Guitart, 1980]
- We want to study the completeness of categories of F-automata
- This is known in the literature for the case $F := \otimes I$ [Ehrig, 1974]: we present a generalization based on a more conceptual approach.
- We formalize some of our results in the Agda proof assistant.

Automata in monoidal categories

- Our setting: automata in a monoidal category $(\mathcal{K}, \otimes, 1)$
- Take two fixed $I, O \in \mathcal{K}$, representing input and output objects.

Definition

A Moore automata $\langle E, d, s \rangle$ in \mathcal{K} is an object E with morphisms d, s:

$$E \overset{d}{\longleftarrow} E \otimes I \;\; ; \;\; E \overset{s}{\longrightarrow} O$$

Definition

A morphism of Moore automata between $\langle E, d, s \rangle$ and $\langle T, d', s' \rangle$ is a morphism $f \colon E \to T$ making the following diagrams commute:

$$E \overset{d}{\longleftarrow} E \otimes I \qquad E \overset{s}{\longrightarrow} O$$

$$f \downarrow \qquad \qquad \downarrow f \otimes I \qquad f \downarrow \qquad \parallel$$

$$T \overset{d}{\longleftarrow} T \otimes I \qquad T \overset{s}{\longrightarrow} O$$

• We denote the category of Moore automata as Moore(I, O).

Mealy automata

Definition

A **Mealy automata** in ${\mathcal K}$ is a span of two morphisms d and s:

$$E \stackrel{d}{\longleftarrow} E \otimes I \stackrel{s}{\longrightarrow} O$$

Definition

A morphism of Mealy automata between $\langle E,d,s\rangle$ and $\langle T,d',s'\rangle$ is a morphism $f\colon E\to T$ making the following diagram commute:

$$E \xleftarrow{d} E \otimes I \xrightarrow{s} O$$

$$f \downarrow \qquad \qquad \downarrow_{f \otimes I} \qquad \parallel$$

$$T \xleftarrow{d'} T \otimes I \xrightarrow{s'} O$$

 \bullet Mealy automata arrange into categories $\mathrm{Mealy}(\mathit{I},\mathit{O})$, which are actually the hom-categories of a bicategory! 1

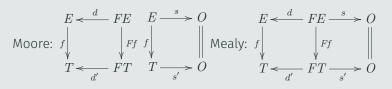
1: [Boccali, Laretto, Loregian, Luneia, 2023]

F-Moore and *F*-Mealy

- A natural generalization: replace $-\otimes I$ in the domain with a generic endofunctor $F: \mathcal{K} \to \mathcal{K}$ acting on the states.
- Idea: imagine *F* as providing an action context for the automaton.

Definition

A morphism of F-automata between $\langle E,d,s\rangle$ and $\langle T,d',s'\rangle$ is a morphism $f\colon E\to T$ making the following diagrams commute:



- We denote the category of *F*-Moore automata by *F*-Moore(*O*).
- A Moore machine is an *F*-Moore machine where $F: \mathcal{K} \to \mathcal{K}$ is the functor $-\otimes I := E \mapsto E \otimes I$.

Examples of F

- Different choices of *F* lead to different notions of automata (e.g., sequential/tree/linear automata) [Adámek-Trnková, 1990].
- In particular, we take into consideration the case where *F* has a right adjoint *R*.

$$E \stackrel{d}{\longleftarrow} FE \stackrel{s}{\longrightarrow} O,$$

$$RE \stackrel{d}{\longleftarrow} E \stackrel{s}{\longrightarrow} RO$$

• In the case where $F:=I\otimes -$, this corresponds to the internal hom R:=[I,-], thus associating a "transition map" to each state.

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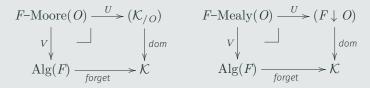
Completeness of categories of categorical automata

- We are interested in the following question:
 - When is the category of F-automata (co)complete?
- Our contribution: a conceptual proof that F-Moore(O) and F-Mealy(O) are (co)complete when K is, based on the theory of 2-pullbacks in Cat and basic facts about limit-preserving functors.
- Proof sketch:
 - 1 Present F-Moore(O) and F-Mealy(O) as 2-pullbacks in Cat.
 - 2 Theorem [Mac Lane, 1998]: if the functors of the pullback satisfy some conditions, then we can compute limits in the pullback.
 - 3 The functors characterizing F-Moore(O) and F-Mealy(O) satisfy the conditions, along with the fact that F is a left adjoint.
 - Φ Hence, the categories are complete when the base category ${\cal K}$ also is.

Characterization of F-Moore and F-Mealy

Theorem

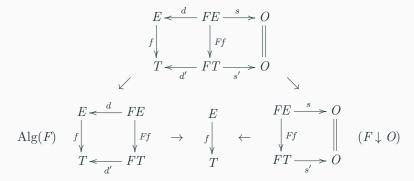
The categories F-Moore(O) and F-Mealy(O) can be characterized as the following strict 2-pullbacks in Cat:



- Alg(F) is the category of algebras of F in K.
- forget is the canonical forgetful functor of F-algebras.
- $(\mathcal{K}_{/O})$ is the slice category of \mathcal{K} over O.
- dom is the forgetful functor of slicecomma categories on the domain.
- $(F \downarrow O)$ is the comma category defined by F and the constant functor on the object O.

Intuition, F-Mealy as pullback

• Intution for the characterization of F-Mealy(O) as pullback in Cat:



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Basic notions on limits

Definition

A functor $F: \mathcal{A} \to \mathcal{B}$ preserves limits of shape $J: \mathcal{I} \to \mathcal{A}$ when, given a limit x in \mathcal{A} , then F(x) is the limit of the composite diagram $\mathcal{I} \xrightarrow{J} \mathcal{A} \xrightarrow{F} \mathcal{B}$.

Definition

A functor $F: \mathcal{A} \to \mathcal{B}$ reflects limits of shape $J: \mathcal{I} \to \mathcal{A}$ when, given a cone x in \mathcal{A} such that F(x) is the limit of the composite diagram $\mathcal{I} \xrightarrow{J} \mathcal{A} \xrightarrow{F} \mathcal{B}$, then x was already a limit of J in \mathcal{A} .

Definition

A functor $F: \mathcal{A} \to \mathcal{B}$ creates limits of shape $J: \mathcal{I} \to \mathcal{A}$ when it both preserves and reflects them.

Pullbacks in Cat and limits

Theorem (Mac Lane 1998, V.6, Ex. 3)

Given a pullback diagram in Cat:

$$\begin{array}{c|c} \mathcal{A} & \xrightarrow{H'} & \mathcal{Y} \\ G' & & \downarrow G \\ \mathcal{X} & \xrightarrow{H} & \mathcal{Z} \end{array}$$

If H creates limits of shape \mathcal{J} and G preserves them, then H' also creates limits of shape \mathcal{J} .

Proposition (Riehl 2016, Prop. 3.3.8)

- The functor **forget** : $Alg(F) \to \mathcal{K}$ **creates** limits.
- The functor $dom : \mathcal{K}_{/O} \to \mathcal{K}$ creates colimits and connected limits.

Proposition (Borceux 1994, Vol. 2, Prop. 4.3.2)

• Since F is a left adjoint, $forget : Alg(F) \to \mathcal{K}$ creates colimits.

Completeness of categories of automata

Theorem

- Let K admit colimits of shape J.
 Then F-Moore(O) and F-Mealy(O) also admit them, and they are computed as in K.
- Let K admit connected limits.
 Then F-Moore(O) and F-Mealy(O) also admit them, and they are computed as in K.

Proof. Immediate using the characterizations of F-Mealy/F-Moore.

Theorem

- Let K admit countable products and pullbacks.
 Then F-Moore(O) and F-Mealy(O) admit products of any finite cardinality (in particular, a terminal object), but they are not computed as in K.
 - → We must define discrete limits explicitly! (Terminal and products)

Behaviour extension

• If $(K, \otimes, 1)$ has countable coproducts preserved by each $- \otimes I$, a Moore automata can be extended to a span:

$$E \stackrel{d^*}{\longleftarrow} E \otimes I^* \stackrel{s^*}{\longrightarrow} O,$$

where $I^* := \sum_{n \geq 0} I^n$ is the freely generated monoid from I, and the morphisms d^*, s^* are defined inductively from components

$$d_n, s_n : E \times I^n \to E, O$$
 for $n \ge 0$.

Similarly, Mealy automata can be extended to $I^+:=\sum_{n\geq 1}I^n$ as

$$E \stackrel{d^+}{\longleftarrow} E \otimes I^+ \stackrel{s^+}{\longrightarrow} O.$$

• Intuition: extend the automata to act on *strings of symbols* instead of single inputs.

Behaviour extension of a F-automata

• An F-Moore automata $\langle E, d, s \rangle$ can be similarly extended; given

$$E \stackrel{d}{\longleftarrow} FE \; ; \; E \stackrel{s}{\longrightarrow} O$$

we define the family of morphisms $s_n: F^nE \to O$ for $n \ge 0$ as the composites

$$\begin{cases} s_0 = E \xrightarrow{s} O \\ s_1 = FE \xrightarrow{d} E \xrightarrow{s} O \\ s_2 = FFE \xrightarrow{F^d} FE \xrightarrow{d} E \xrightarrow{s} O \\ s_n = F^n E \xrightarrow{F^{n-1} d} F^{n-1} E \xrightarrow{s} \cdots \xrightarrow{FFd} FFE \xrightarrow{Fd} FE \xrightarrow{d} E \xrightarrow{s} O \end{cases}$$

• In our assumption where $F \dashv R$, each map is equivalent to its mate

$$\frac{s_n: F^n E \to O}{\overline{s}_n: E \to R^n O} \text{ for } n \ge 0$$

obtained by iterating the adjunction structure.

Skip and behaviour maps

• Each morphism obtained like this

$$\bar{s}_n: E \to R^n O$$

is called the n-th skip map, since it gives the dynamics of a state after skipping n input steps.

• In case \mathcal{K} has countable products, the family of all n-th skip maps $(s_n \mid n \in \mathbb{N}_{\geq 0})$ is equivalent to a single map

$$beh_{\mathsf{E}}: E \to \prod_{n>0} R^n O$$

called the behaviour map of the automata $\mathbf{E} := \langle E, d, s \rangle$.

Terminal object

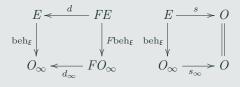
• The behaviour map has a specific universal property:

Theorem (Terminal object of F-Moore)

The category F-Moore has a terminal object

$$\mathfrak{o} = \langle O_{\infty}, s_{\infty}, d_{\infty} \rangle$$
, where $O_{\infty} = \prod_{n \geq 0} \mathbb{R}^n O$.

Explicitly, for any other F-Moore automata $\mathbf{E}:=\langle E,d,s\rangle$, the behaviour map $\mathrm{beh}_{\mathsf{E}}:E\to O_\infty$ is the unique morphism making the following diagrams commute:



Terminal object, explicitly

ullet The terminal object O_{∞} in a category of machines tends to be "big", since it can be obtained by Adámek's theorem as the terminal coalgebra for the functor

$$A \mapsto O \times RA$$
 for F -Moore(O), $A \mapsto RO \times RA$ for F -Mealy(O).

• The morphism d_{∞} is defined using the universal property of the product by combining the family $(d_i \mid i \geq 0)$, given as

$$d_i := \frac{\prod_{n \geq 0} R^n O \xrightarrow{\pi_{i+1}} R^{i+1} O}{F(\prod_{n \geq 0} R^n O) \xrightarrow{\bar{\pi}_{i+1}} R^i O}, \quad d_{\infty} : F(\prod_{n \geq 0} R^n O) \to \prod_{n \geq 0} R^n O$$

and s_{∞} is simply the first projection:

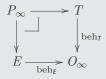
$$s_{\infty} := \prod_{n \geq 0} R^n O \xrightarrow{\pi_0} O$$

• Intuition: d_{∞} advances the behaviour by one step, and s_{∞} outputs.

Products in F-Moore and F-Mealy

Theorem (Products of *F*-automata)

Given F-Moore automata $\mathbf{E} := \langle E, d, s \rangle, \mathbf{T} := \langle T, d', s' \rangle$, the pullback



exhibits the carrier of an F-Moore automata $\mathfrak{p} := \langle P_{\infty}, d_P, s_P \rangle$ that has the universal property of the **product** of **E** and **T** in F-Moore(O).

• Intuition: P_{∞} is the set of pairs of states $(\alpha, \beta) \in E \times T$ such that for every string of inputs $\mathrm{beh}_{\mathsf{E}}(\alpha) = \mathrm{beh}_{\mathsf{T}}(\beta)$, i.e., their behaviour coincides: P_{∞} corresponds to a *bisimulation* object.

Adjoints to behaviour functors

• Our approach generalizes the one of Naudé [1977, 1979]:

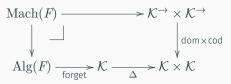
Definition

Call an endofunctor $F: \mathcal{K} \to \mathcal{K}$ an input process if the forgetful functor $U: \mathrm{Alg}(F) \to \mathcal{K}$ has a left adjoint G; in simple terms, an input process allows to define free F-algebras.

 Naudé [1977, 1979] concentrates on building an adjunction between a category of machines and a category of their behaviours

$$L: \operatorname{Beh}(F) \xrightarrow{} \operatorname{Mach}(F) : E$$

where Mach(F) is the category obtained from the pullback



and Beh(F) is a certain comma category on G.

Adjoints to behaviour functors

 This theorem is conceptual enough to carry over to any category of automata that can be presented as strict 2-pullback in Cat of sufficiently well-behaved functors.

Theorem

There exist functors B and L, as follows:

$$B: Alg(F)_{/\langle O_{\infty}, d_{\infty} \rangle} \xrightarrow{} F-Moore(O): L$$

where $\langle O_{\infty}, d_{\infty} \rangle$ is the terminal (behaviour) F-algebra given.

Theorem

This is part of a longer chain of adjoints obtained as follows:

$$\mathcal{K}_{/O_{\infty}} \xrightarrow{\tilde{G}} \operatorname{Alg}(F)_{/(O_{\infty},d_{\infty})} \xrightarrow{L} F\text{-Moore}(O),$$

where we denote with $\tilde{G}: \mathcal{K}_{/UA} \leftrightarrows \mathcal{H}_{/A}: \tilde{U}$ the "local" adjunction obtained from $G: \mathcal{K} \leftrightarrows \mathcal{H}: U$, with $\tilde{U}(FA, f: FA \to A) = Uf$.

Agda formalization

 We have formalized the more technical parts of our work in Agda, a dependently typed programming language and proof assistant.



- Formalization work:
 - Characterization of F-Moore(O)/F-Mealy(O) as pullbacks in Cat.
 - Products and terminal objects in F-Moore(O), explicitly.
 - Adjoints to behaviour functors, generalizing Naudé's approach.
 - Mealy(I, O) are the hom-categories of the bicategory **Mealy**.
- We use the agda-categories library as a foundation to capture the basic notions of category theory.
- (Almost 2000 lines of code!)
- Formalization is freely available online:

https://github.com/iwilare/categorical-automata

Conclusion and Future work

- Characterizing categories of structures as *composition of simpler* categories can be a useful technique to compute limits.
- Bigger picture: the technology of category-theoretic approaches is rapidly shifting towards 2-dimensional categories as foundations for complex systems [Spivak et al. 2019], [Myers, 2021]
- Generalize other aspects of automata theory from the point of view of higher category theory (e.g. Krohn-Rhodes theorem).
- Formalizing these results in a proof assistant might pave the way for more concrete applications, where proofs act as programs to produce and convert automata in a provably correct way.

Thank you!